A Variational Analysis of Dielectric Waveguides by the Conformal Mapping Technique

RUEY-BEEI WU AND CHUN HSIUNG CHEN

Abstract —**The variational formulation together with the finite-element method is a well-established technique for the solution of a dielectric waveguide. One common difficulty is the handling of the problem with the infinite extent of the electromagnetic fields in the transverse plane. In this paper, the conformal mapping technique is employed to improve the modeling of the region exterior to the guides; hence it may give more accurate results for the modes near the cutoff region. Also included are the numerical results for rectangular, strip, and channel waveguides to demonstrate the applications of the proposed technique.**

I. Introduction

IN MILLIMETER- AND optical-wave spectra, various applications of dielectric waveguides have been suggested (for instance, as a directional coupler [1], a phase shifter [2], [3], and a channel-dropping filter [4]). In the design of these structures, it is important to calculate the propagation constants and the field patterns of the waveguide. Some guiding structures are so important as to warrant specialized methods adapted to their needs. Typical examples include microstriplines, optical fibers, and rectangular waveguides. For other guides with complicated geometry and complex media, the finite-element method is probably the most flexible and versatile one for analysis.

In general, the variational equations for dielectric waveguides and close-type waveguides [5] are essentially the same in mathematics. Since the fields in dielectric waveguides extend to infinity, the integration in the variational equation must cover the whole transverse plane. The variational methods employ exterior region basis functions with exponentially decaying parameters which need to be optimized [6]–[8]. The variational reaction theory obtains a variational equation with integration in the finite region by properly absorbing the radiation condition and the continuity conditions, and, hence, needs some mode searching scheme [9].

On the other hand, the finite-element methods employ local basis functions and take care of the modeling of the infinite transverse extent of the fields. The most common solution is a simple truncation of the exterior fields by imposing metallic walls at a large distance from the guide [10], [11]. Another and perhaps better solution is the use of an infinite element with an empirical decaying parameter prescribed [12], [13] or with some special basis functions which need to be considered separately [14].

In this paper, a new and rigorous approach is proposed by conformally mapping the whole transverse plane to a suitable finite region. The governing variational equation then remains almost invariant and can thus be solved directly by the conventional finite-element method [15], [16]. Since the fields exterior to the guide are, in general, more insignificant than the interior ones, this proposed method causes no difficulty when the exterior region is conformally condensed. Therefore, the problem may be more efficiently tackled in the new transformed finite region.

Manuscript received October 29, 1984; revised March 18, 1985.
The authors are with the Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan, R.O.C.

II. Method of Analysis

A. Variational Formulation

Consider a uniform dielectric waveguide of arbitrary cross section and with an inhomogeneous medium (Fig. 1). Let the relative permittivity and permeability be $\epsilon_r(x, y)$ and $\mu_r(x, y)$, respectively. It is well known that the propagating modes of a dielectric waveguide are generally hybrid. Both axial components E_z and H_z are required to characterize all the field components. Thus, the governing variational equation for this structure can be written as [5]

$$\begin{aligned} &\delta I = 0 \\ &I = \iint_\Omega dx\,dy\,\frac{1}{\epsilon_r\mu_r - n_e^2}\left(\epsilon_r|\nabla_t E_z|^2 + \mu_r\eta_0^2|\nabla_t H_z|^2\right. \\ &\qquad \left. + 2n_e\eta_0\hat{z}\cdot(\nabla_t E_z \times \nabla_t H_z)\right) \\ &\qquad - k_0^2\iint_\Omega dx\,dy\left(\epsilon_r|E_z|^2 + \mu_r\eta_0^2|H_z|^2\right) \qquad (1) \end{aligned}$$

where the integration region Ω should cover the whole transverse plane. Here, k_0 is the wavenumber in free space, η_0 is the characteristic impedance of free space, and n_e^2 is the effective dielectric constant which relates to the propagation constant β by

$$n_e = \beta/k_0. \qquad (2)$$

B. Conformal Mapping

The conformal mapping technique is a useful tool in the analysis of static field problems. Its application to the time-harmonic waveguide problem will be presented in this

0018-9480/85/0800-0681$01.00

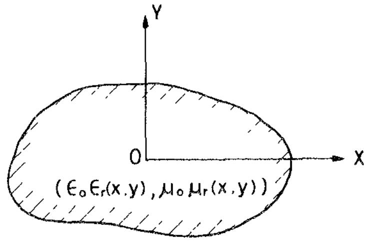


Fig. 1. Geometry of an arbitrarily-shaped inhomogeneous dielectric waveguide which is uniform in the z-direction.

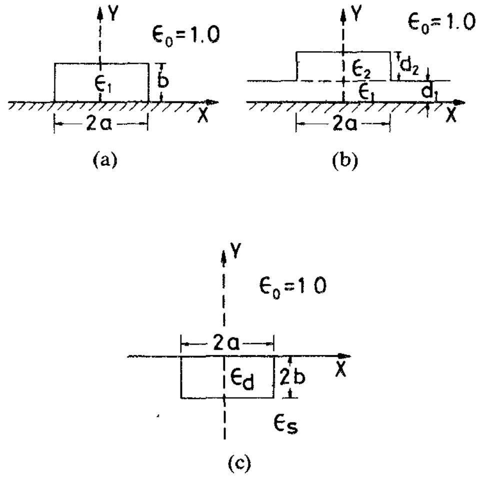


Fig. 2. Three dielectric waveguides which are symmetric with respect to the $y-z$ plane: (a) image guide, (b) strip guide, and (c) channel guide.

section. Let the relation between the original coordinate $w=(x, y)$ and the new coordinate $w'=(x', y')$ be defined by an analytic complex function

$$w'=f(w). \tag{3}$$

By this conformal transformation (3), the variational equation (1) in the new coordinate system thus becomes

$$\begin{aligned} I=\iint_{\Omega'} dx'\,dy' \frac{1}{\epsilon_r\mu_r-n_e^2}\Big(&\epsilon_r|\nabla_t' E_z|^2+\mu_r\eta_0^2|\nabla_t' H_z|^2 \\ &+2n_e\eta_0\hat{z}\cdot(\nabla_t' E_z\times\nabla_t' H_z)\Big) \\ &-k_0^2\iint_{\Omega'} dx'\,dy'\,|J|\big(\epsilon_r|E_z|^2+\mu_r\eta_0^2|H_z|^2\big) \end{aligned} \tag{4}$$

where the Jacobian $|J|$ is related to the complex function by

$$|J|=\left|\frac{dw}{dw'}\right|^2_{w'=(x',y')}. \tag{5}$$

Equations (4) and (1) mathematically have the same form except the last term. This simplicity may be attributed to the angular invariance of the conformal mapping.

Let us consider the guiding structures which are symmetric with respect to the $y-z$ plane (Fig. 2). This $y-z$ plane can be regarded as an electric or a magnetic wall when odd or even modes are considered. Therefore, it is sufficient to solve the problem in the $x\geq 0$ plane. It is well known that this half plane can be conformally mapped into a unit circle by the linear fractional transformation [17]

$$w'=f(w)=\frac{w-1}{w+1}. \tag{6}$$

The Jacobian of the transformation is thus

$$|J|=\left|\frac{2}{(1-w')^2}\right|^2=\frac{4}{\left((1-x')^2+y'^2\right)^2}. \tag{7}$$

Though the Jacobian is singular at $w'=1$, where $|w|$ tends to infinity, the integrand in the last term of (4) still remains finite and regular since the fields E_z and H_z for the guided modes monotonically vanish there. For leaky modes where the exterior fields are oscillatory, it is difficult to choose proper basis functions for the elements containing the point $w'=1$ since the integrand now is finite but irregular there. However, the method would still give reasonable results if more divisions are employed and the exterior fields decay very fast so that the error caused by this irregularity is negligible.

C. Finite-Element Method

Since the integration range Ω' in (4) is finite, it can be solved by the conventional finite-element method. We first discretize the entire region Ω' into a finite number of subregions, called elements. As an example, let us consider the rectangular image guide which also possesses another symmetry with respect to the $x-z$ plane (Fig. 2(a)). Fig. 3 shows typical elements in both the original and the new coordinate systems. In each element, the field ϕ, which denotes E_z or H_z, is expressed as

$$\phi(x', y')=\sum_i \phi_i B_i(\xi,\eta) \tag{8}$$

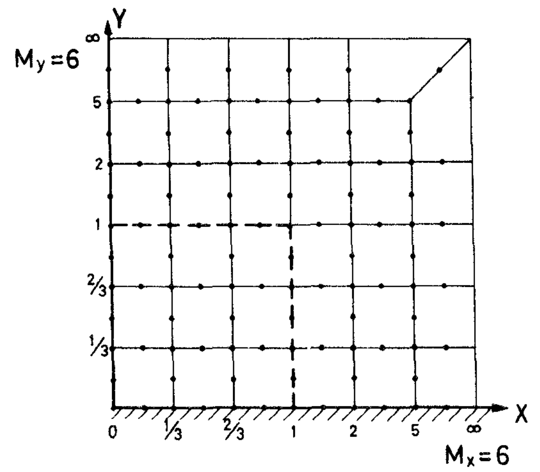
where ϕ_i is the nodal unknown and B_i is a suitable shape function [15]. Also the global coordinate (x', y') of a node is isoparametrically related to the local coordinate (ξ, η) by [15]

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\sum_i \begin{bmatrix} x_i' \\ y_i' \end{bmatrix} B_i(\xi,\eta). \tag{9}$$

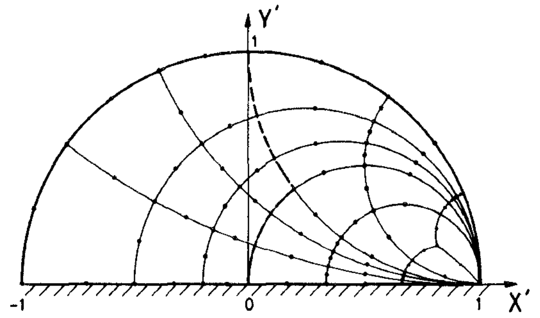
Then, we have to calculate the integrals contributed from each element. To take the inhomogeneity and the Jacobian into consideration, we use the Gaussian quadratic formula [15] for integration. By assembling the element integrals and applying the Ritz procedure, we finally obtain the matrix equation

$$[A][\Phi]=k_0^2[B][\Phi] \tag{10}$$

where $[\Phi]$ is the column vector corresponding to the nodal unknowns, while $[A]$ and $[B]$ are known matrices which are of the banded type. Though the matrix $[A]$ is not positive-definite, (10) can still be effectively solved by searching for k_0 such that the determinant of $([A]-k_0^2[B])$ vanishes [16].



(a)



(b)

Fig. 3. Typical subdivision elements for image guide in (a) original and (b) transformed coordinate systems. Here, M_x and M_y are the number of elements in the x- and y-directions, respectively. The dashed line represents the actual boundary of image guide.

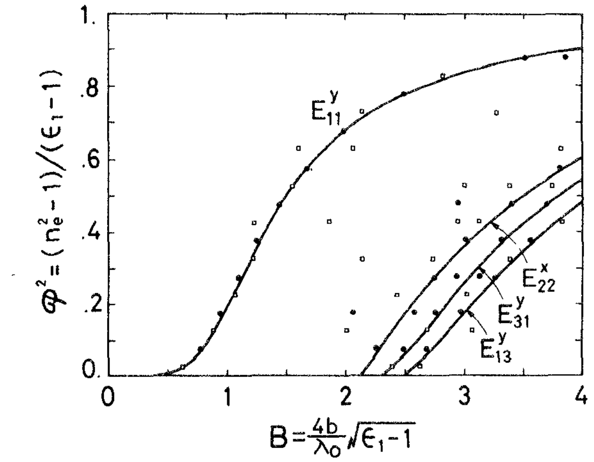


Fig. 4. Computed results for image guide (Fig. 2(a)) with $a/b=1$, and $\epsilon_1=2.25$. Here, • and □ represent the results for $M_x=M_y=6$ and $M_x=M_y=3$, respectively. The solid curves are Goell's results.

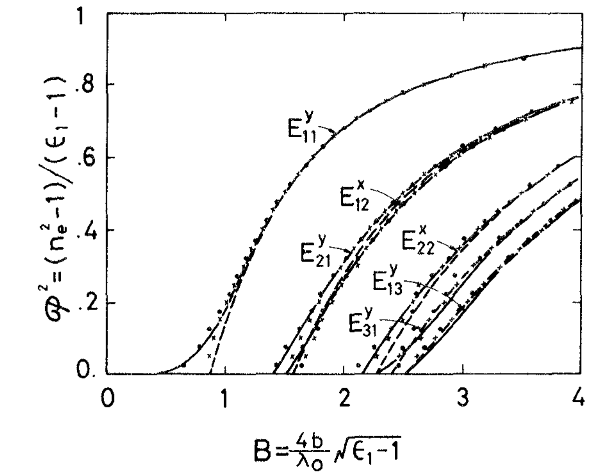


Fig. 5. Dispersion curves for first six guided modes of image guide with $a/b=1$, $\epsilon_1=2.25$, and $M_x=M_y=6$. The dot and cross marks represent the finite-element results with conformal mapping technique and simple truncation method, respectively, while the solid and dashed curves are Goell's and Marcatili's results, respectively.

III. Numerical Results

In this section, several guiding structures will be analyzed using the method described. We first consider the rectangular image guide as shown in Fig. 2(a). For the modes with E_z symmetric (and H_z anti-symmetric) to the $y-z$ plane, their dispersion curves are shown in Fig. 4. As compared with Goell's results [18], the computed propagation constants of the dominant mode E^y_{11} are accurate even with $M_x \times M_y = 3\times 3 = 9$ elements. However, more subdivision elements are required to give accurate results for the higher modes. Also shown in the figure are the constants of the spurious, nonphysical modes. It is interesting to note that, with more subdivision elements, the spurious modes are fewer in a prescribed range of B (e.g., $0 \leqslant B \leqslant 4$) and, therefore, can be eliminated more easily.

The other modes with E_z anti-symmetric (and H_z symmetric) to the $y-z$ plane are obtained by imposing an electric wall over the $y-z$ plane. The dispersion curves for the first six guided modes of the image guide are shown in Fig. 5. Compared with the Marcatili's approximation [19], the present method can give more accurate results for the modes near cutoff and for the E^x_{12} and E^y_{21} modes, which are nearly degenerate. The cross marks in the figure are the results obtained by the simple truncation method with the artificial electric walls imposed at a distance twice the dimension of the guide. It is shown that the present method gives a significant improvement for the modes whose fields are not well confined by the guide boundary.

This method is also applicable to the slab-coupled waveguide such as the strip guide shown in Fig. 2(b). Fig. 6 shows the results for the three guided modes of this structure. Here, the solid curve for the dominant mode is almost identical to the one obtained by McLevige *et al.* [20], but a little different from the one obtained by Ikeuchi *et al.* [21]. In the latter literature, the finite-element method has also been adopted but using an iterative algorithm to handle the unbounded exterior region.

We next consider the channel guide (Fig. 2(c)) where only one symmetry with respect to the $y-z$ plane exists. The conformal mapping defined by (6) can also be applied directly, but more subdivision elements in the y-direction are required. The results for the first eight modes are

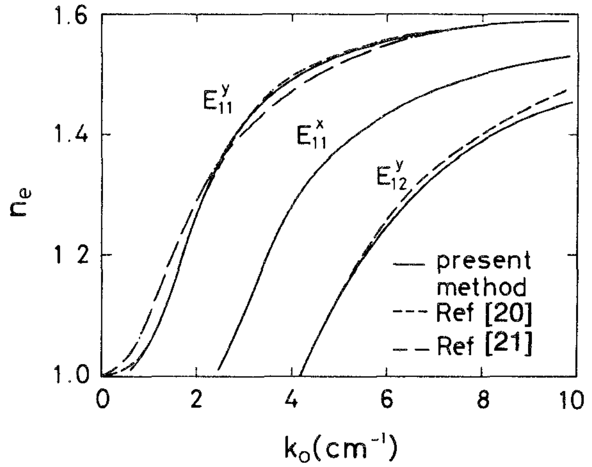


Fig. 6. Dispersion curves for three guided modes of strip guide (Fig. 2(b)) with $2a = 0.65$ cm, $d_1 = 0.5$ cm, $d_2 = 0.32$ cm, $\epsilon_1 = 2.62$, $\epsilon_2 = 2.55$, and $M_x = 6$, $M_y = 7$.

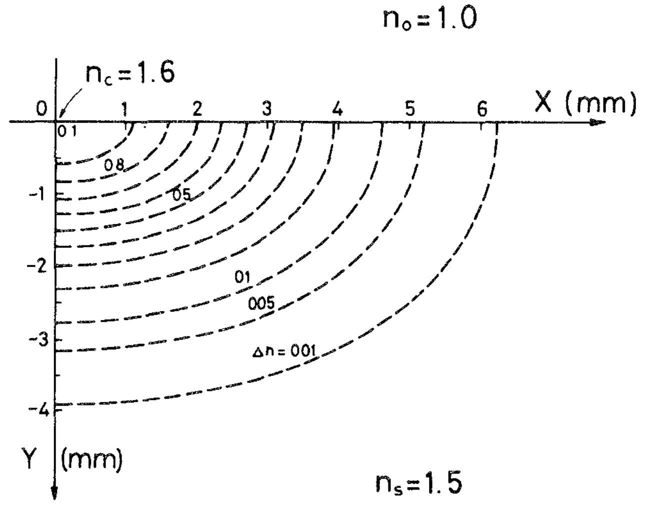


Fig. 8. Profile of index change (Δn) for inhomogeneous channel guide.

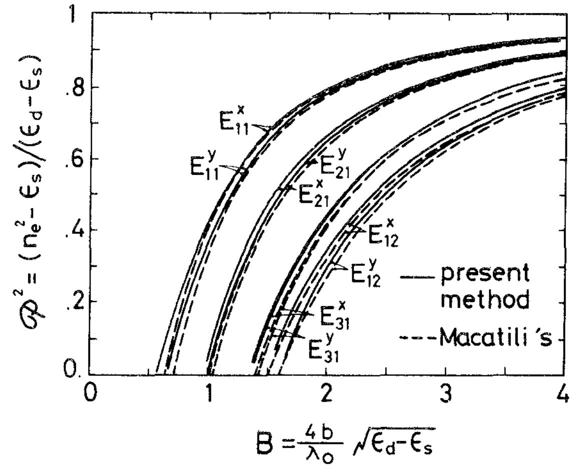


Fig. 7. Dispersion curves for first eight guided modes of channel guide (Fig. 2(c)) with $a/b = 2$, $\epsilon_d = 2.56$, $\epsilon_s = 2.25$, $M_x = 6$, and $M_y = 9$.

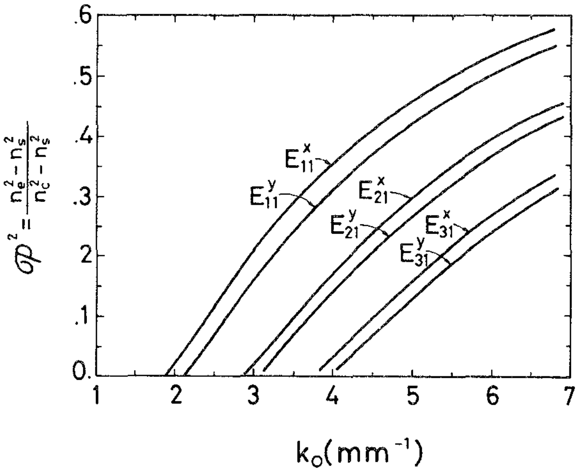


Fig. 9. Results for first six guided modes of inhomogeneous channel guide.

shown in Fig. 7. They show good agreement with the curves obtained by Marcatili's approximate formula. As compared with the image guide and strip guide, the channel guide has its lowest mode polarized in the x-direction instead.

Finally, the present method is applied to analyze the channel guide consisting of an inhomogeneous substrate. The refractive index in the substrate is assumed to be

$$n(x, y) = n_s + \frac{(n_c - n_s)}{2\cdot \text{erf}(W_x/D_x)} \cdot \left[\text{erf}\left(\frac{W_x + x}{D_x}\right) + \text{erf}\left(\frac{W_x - x}{D_x}\right)\right] \cdot \exp\left(-\frac{y^2}{D_y^2}\right) \quad (11)$$

where "erf" means the error function. Fig. 8 shows the profile of the index change for the guide with parameters: $n_c = 1.6$, $n_s = 1.5$, $W_x = 2.5$ mm, $D_x = 2.25$ mm, and $D_y = 1.82$ mm. Here one still has the symmetry with respect to the $y-z$ plane. Fig. 9 shows the computed dispersion curves for the first six guided modes.

IV. Conclusions

In this paper, the vector variational finite-element method in conjunction with the conformal mapping technique has been established for analyzing problems which extend to infinity. The method has been applied to handle a wide variety of dielectric waveguide structures which are symmetric with respect to the vertical center plane. One major implication of this work is that more accurate results for the guided mode near cutoff can be obtained due to the proper treatment of the exterior field.

The efficiency of this method relies much of what conformal mapping function is adopted. In particular, by choosing a suitable conformal mapping, some complicated problems can be solved very efficiently. The related works, such as the study of a dielectric waveguide coupler, are in progress and will be reported in the future.

References

[1] J. A. Paul and Y. W. Chang, "Millimeter-wave image-guide integrated passive devices," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-26, pp. 751–754, Oct. 1978.

[2] H. Jacobs and M. M. Chrepta, "Electronic phase shifter for millimeter-wave semiconductor dielectric integrated circuits," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-22, pp. 411–417, Apr. 1974.
[3] C. H. Lee, P. S. Mak, and A. P. DeFonzo, "Optical control of millimeter-wave propagation in dielectric waveguides," *IEEE J. Quantum Electron.*, vol. QE-16, pp. 277–288, Mar. 1980.
[4] T. Itanami and S. Shindo, "Channel-dropping filter for millimeter-wave integrated circuits," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-26, pp. 759–764, Oct. 1978.
[5] P. Daly, "Hybrid-mode analysis of microstrip by finite-element methods," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 19–25, Jan. 1971.
[6] M. Ohtaka, M. Matsuhara, and N. Kumagai, "Analysis of the guided modes in slab-coupled waveguides using a variational method," *IEEE J. Quantum Electron.*, vol. QE-12, pp. 378–382, July 1976.
[7] S. Akiba and H. A. Haus, "Variational analysis of optical waveguides with rectangular cross section," *Appl. Opt.*, vol. 21, pp. 804–808, Mar. 1982.
[8] I. V. Lindell and M. I. Oksanen, "Transversely anisotropic optical fibers: Variational analysis of a nonstandard eigenproblem," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-31, pp. 736–744, Sept. 1983.
[9] R. B. Wu and C. H. Chen, "Variational reaction theory for dielectric waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-33, pp. 477–483, June 1985.
[10] C. Yeh, S. B. Dong, and W. Oliver, "Arbitrarily shaped inhomogeneous optical fiber or integrated optical waveguides," *J. Appl. Phys.*, vol. 46, pp. 2125–2129, May 1975.
[11] N. Mabaya, P. E. Lagasse, and P. Vandenbulcke, "Finite-element analysis of optical waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-29, pp. 600–605, June 1981.
[12] C. Yeh, K. Ha, S. B. Dong, and W. P. Brown, "Single-mode optical waveguides," *Appl. Opt.*, vol. 18, pp. 1490–1504, May 1979.
[13] B. M. A. Rahman and J. B. Davies, "Finite-element analysis of optical and microwave waveguide problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-32, pp. 20–28, Jan. 1984.
[14] P. Bettess, "Infinite elements," *Int. J. Num. Meth. Eng.*, vol. 11, pp. 53–64, 1977.
[15] O. C. Zienkiewicz, *The Finite Element Method*. New York: McGraw-Hill, 1977.
[16] K. J. Bathe and E. L. Wilson, *Numerical Methods in Finite Element Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1976.
[17] E. Kreyszig, *Advanced Engineering Mathematics*, 3rd ed. New York: Wiley, 1972.
[18] J. E. Goell, "A circular-harmonic computer analysis of rectangular dielectric waveguides," *Bell Syst. Tech. J.*, vol. 48, pp. 2133–2160, Sept. 1969.
[19] E. A. J. Marcatili, "Dielectric rectangular waveguide and directional coupler for integrated optics," *Bell Syst. Tech. J.*, vol. 48, pp. 2071–2102, Sept. 1969.
[20] W. V. McLevige, T. Itoh, and R. Mittra, "New waveguide structures for millimeter-wave and optical integrated circuits," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 788–794, Oct. 1975.
[21] M. Ikeuchi, H. Sawami, and H. Niki, "Analysis of open-type dielectric waveguide by the finite-element iterative method," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-29, pp. 234–239, Mar. 1981.

Ruey-Beei Wu was born in Tainan, Taiwan, Republic of China, on October 27, 1957. He received the B.S.E.E. degree from National Taiwan University, Taipei, Taiwan, in 1979. Since 1981, he has been studying toward the Ph.D. degree at the same university.

In 1982, he joined the faculty of the Department of Electrical Engineering, National Taiwan University, where he is now an Instructor. He is currently engaged in the area of numerical methods to electromagnetic-field problems.

Chun Hsiung Chen was born in Taipei, Taiwan, Republic of China, on March 7, 1937. He received the B.S.E.E. degree from National Taiwan University, Taipei, Taiwan, in 1960, the M.S.E.E. degree from National Chiao Tung University, Hsinchu, Taiwan, in 1962, and the Ph.D. degree in electrical engineering from National Taiwan University in 1972.

In 1963, he joined the faculty of the Department of Electrical Engineering, National Taiwan University, where he is now a Professor. Since August 1982, he has also been the Chairman of the same department. In 1974, he was a Visiting Researcher for one year at the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley. His areas of interest include antenna and waveguide analysis, propagation and scattering of waves, and numerical techniques in electromagnetics.